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Alternative bi-Hamiltonian structures for WDVV equations of associativity

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Abstract. The WDVV equations of associativity in two-dimensional topological field theory are completely integrable third-order Monge–Ampère equations which admit bi-Hamiltonian structure. The time variable plays a distinguished role in the discussion of Hamiltonian structure, whereas in the theory of WDVV equations none of the independent variables merits such a distinction. WDVV equations admit very different alternative Hamiltonian structures under different possible choices of the time variable, but all these various Hamiltonian formulations can be brought together in the framework of the covariant theory of symplectic structure. They can be identified as different components of the covariant Witten–Zuckerman symplectic 2-form current density where a variational formulation of the WDVV equation that leads to the Hamiltonian operator through the Dirac bracket is available.

1. Equations of associativity

In two-dimensional (2D) topological field theory Witten [1, 2] has shown that model independent n-point correlation functions follow recursively from the 2- and 3-point correlation functions that serve to define a non-degenerate flat metric and structure functions of a Frobenius algebra. These are the principal objects in this theory and they can be expressed as third derivatives of a generating function

$$c_{ijk} = \frac{\partial^3 F}{\partial t^i\, \partial t^j\, \partial t^k} \tag{1}$$

where F is the free energy [3]. One of the independent variables, say t^1, is singled out to serve in the definition of the metric

$$c_{1ij} \equiv \eta_{ij} \tag{2}$$

which is assumed to be non-degenerate. The structure functions satisfy a Frobenius algebra and the conditions of associativity

$$c_{mi[j}\, \eta^{mn}\, c_{k]ln} = 0 \tag{3}$$

result in third-order Monge–Ampère equations which are known as WDVV equations. Indices enclosed by square parentheses are skew-symmetrized.

Dubrovin [4] has given a systematic account of WDVV equations of associativity which are completely integrable systems. They admit bi-Hamiltonian structure [5, 6] which also provides proof of their complete integrability through the theorem of Magri [7]. However, in the discussion of Hamiltonian structure time, which can be identified to be any one of

t^i, $i \neq 1$, is necessarily singled out whereas in the general theory of WDVV equations no such distinction exists. To give an example, consider the free energy

$$F_1 = \tfrac{1}{2}(t^1)^2 t^2 + \tfrac{1}{2} t^1 (t^3)^2 + f(t^2, t^3) \tag{4}$$

that through the identification

$$t^2 = x \qquad t^3 = t \tag{5}$$

results in Dubrovin's equation of associativity

$$f_{ttt} + f_{xxx} f_{ttx} - f_{txx}^2 = 0. \tag{6}$$

By a trivial interchange in the roles of t and x, alternatively if in place of (5) we make the identification

$$t^2 = t \qquad t^3 = x \tag{7}$$

then we are led to

$$f_{ttt} f_{txx} - f_{ttx}^2 + f_{xxx} = 0 \tag{8}$$

which should be the same equation of associativity as there is no distinction between the independent variables t and x that stems from 2D topological field theory itself. However, from the point of view of Hamiltonian structure, equations (6) and (8) are radically different. The bi-Hamiltonian structure of (6) which is based on the results of [8] and [9] was given in [5], and in this paper we present the bi-Hamiltonian structure of (8). We find that the Hamiltonian operators appropriate to (8) are quite different from those obtained for (6), however, and we further show that they can be identified as different aspects of the same structure when we consider them in the framework of the covariant Witten–Zuckerman [10] formulation of symplectic structure. Considerations of covariant symplectic structure take the variational formulation as their starting point. For WDVV equations only the Lagrangian that yields their second Hamiltonian structure through Dirac's theory of constraints [11] is known, and therefore this part of our discussion will necessarily be restricted. We show that the second Hamiltonian operators for equations (6) and (8) are simply the inverse of the t and x components of the Witten–Zuckerman symplectic current 2-form for only one of these equations.

Exactly the same situation holds for the WDVV equation that follows from the free energy

$$F_2 = \tfrac{1}{2}(t^1)^2 t^3 + \tfrac{1}{2} t^1 (t^2)^2 - \tfrac{1}{2}(t^1)^2 t^2 + f(t^2, t^3) \tag{9}$$

which through the identification (5) leads to

$$f_{ttt} + f_{ttt} f_{xxx} - f_{ttx} f_{txx} + f_{ttt} f_{txx} - f_{xtt}^2 + f_{xxx} f_{xtt} - f_{xxt}^2 = 0 \tag{10}$$

and the bi-Hamiltonian structure of this equation was presented in [6]. With the same free energy the opposite identification (7) results in

$$f_{ttt} f_{xxx} - f_{ttx} f_{txx} + f_{xxx} f_{xtt} - f_{xxt}^2 + f_{ttt} f_{txx} - f_{xtt}^2 + f_{xxx} = 0 \tag{11}$$

as the WDVV equation of associativity. Once again we find that the bi-Hamiltonian structure of equation (11) is different from the earlier results obtained for (10) but they can be recognized as different components of the closed conserved current 2-form in the covariant formulation of symplectic structure.

2. System of evolution equations

In earlier literature [12] Monge–Ampère equations which are second-order partial differential equations consisting of an appropriate sum of linear terms and the Hessian were discussed in the framework of linear equations proper. The reason for this lies in the fact that the initial value problem for hyperbolic Monge–Ampère equations is identical to that of the second-order linear equation [13] in spite of the severe nonlinearities introduced by the Hessian. WDVV equations of associativity are of third order but they consist of a sum of linear terms and the minor determinants of the third-order Hankelian

$$H(u,t,x) = \text{detminor}\begin{Bmatrix} f_{ttt} & f_{ttx} & f_{txx} \\ f_{ttx} & f_{txx} & f_{xxx} \end{Bmatrix} \tag{12}$$

closely analogous to the case of Monge–Ampère equations. Therefore, it seems reasonable to conjecture that the initial value problem for WDVV equations is qualitatively the same as the third-order linear equation. This is particularly important in the discussion of the Hamiltonian structure of WDVV equations where we use the techniques originally developed for real Monge–Ampére equations [14]. For this purpose we need to cast the WDVV equation into the form of a triplet of evolution equations. Introducing the usual auxiliary variables

$$a = f_{xxx} \qquad b = f_{xxt} \qquad c = f_{xtt} \tag{13}$$

we have in place of the WDVV equation (8) the set of evolution equations

$$\begin{aligned} &a_t = b_x \qquad b_t = c_x \\ &c_t = e_x \qquad e \equiv \frac{c^2 - a}{b} \end{aligned} \tag{14}$$

which consists of equations of hydrodynamic type [15]. We note that such a decomposition of equation (8) is not unique but this particular choice of auxiliary variables is useful because the Hamiltonian structure of equations of hydrodynamic type is a well developed subject. In the case of equations (14) the system is linearly degenerate and it will be necessary to use the results of Ferapontov [16] on the Hamiltonian structure of non-diagonizable equations of hydrodynamic type for which Riemann invariants do not exist. From equations (14) it is manifest that a, b, c are conserved densities and there are two others

$$\begin{aligned} &\mathcal{P} = bD^{-1}c \\ &\mathcal{E} = cD^{-1}bD^{-1}c + D^{-1}aD^{-1}b \end{aligned} \tag{15}$$

which consist of momentum and energy. There exist no further conserved quantities of hydrodynamic type as equations (14) are non-diagonizable. Here and in the following D^{-1} stands for the inverse of the total derivative operator $D = \mathrm{d}/\mathrm{d}x$ and its precise definition can be found in [17].

3. Bi-Hamiltonian structure

We first state the principal result we present about equation (8).

Theorem 1. Equations (14) can be written as a bi-Hamiltonian system

$$J_0\,\delta H_0 = J_1\,\delta H_1 \tag{16}$$

where δ denotes the variational derivative with respect to a, b, c and the Hamiltonian operators given by

$$J_0 = \begin{pmatrix} 2aD + 2Da & 3bD + b_x & 2cD \\ 3bD + 2b_x & cD + Dc & eD \\ 2Dc & De & -3D \end{pmatrix} \tag{17}$$

$$J_1 = \begin{pmatrix} -D^3 & 0 & 0 \\ 0 & 0 & D^2 \frac{1}{b} D \\ 0 & D\frac{1}{b}D^2 & D\frac{1}{b}D\frac{c}{b}D + D\frac{c}{b}D\frac{1}{b}D \end{pmatrix} \tag{18}$$

are compatible so that according to Magri's theorem [7] we have a completely integrable system.

The Hamiltonian functions that yield the equations of motion are integrals of the densities $\mathcal{H}_0 = b$ and $\mathcal{H}_1 = \mathcal{E}$, respectively.

4. Spectral problem

The general framework for casting WDVV equations into the form of a spectral problem was given by Dubrovin [4]. In the case of equation (8), or rather equations (14), these considerations yield the Lax pair

$$\Psi_x = zA\Psi \qquad \Psi_t = zB\Psi$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ a & 0 & b \\ b & 0 & c \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ b & 0 & c \\ c & 1 & e \end{pmatrix} \tag{19}$$

where z is the spectral parameter and the compatibility conditions

$$A_t = B_x \qquad [A, B] = 0$$

are satisfied by virtue of equations (14). From the compatibility conditions it follows that the roots u^1, u^2, u^3 of the characteristic equation

$$\det(\lambda I - A) = \lambda^3 - c\lambda^2 - a\lambda + ac - b^2 = 0 \tag{20}$$

are conserved Hamiltonian densities for equations (14).

5. First Hamiltonian structure

The easiest way to find the first Hamiltonian structure of equations (14) is to transform to a set of new dependent variables which consist of the roots of the cubic (20) because the Hamiltonian operator then assumes the form of a first-order homogeneous operator with constant coefficients. The relationship between the roots of this cubic to the hydrodynamic variables a, b, c is given by

$$\begin{aligned} a &= -\beta & \alpha &= u^1 + u^2 + u^3 \\ b &= \mp\sqrt{\gamma - \alpha\beta} & \beta &= u^1u^2 + u^2u^3 + u^3u^1 \\ c &= \alpha & \gamma &= u^1u^2u^3 \end{aligned} \tag{21}$$

according to the formulae of Viète. In these variables the equations of motion (14) assume the form

$$u^i_t = \left(\frac{(u^i)^2 + \beta}{\sqrt{\gamma - \alpha\beta}} \right)_x \qquad i = 1, 2, 3 \tag{22}$$

of Hamilton's equations with the Hamiltonian operator

$$J_0 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} D \tag{23}$$

and the Hamiltonian density

$$\mathcal{H}_0 = b = \sqrt{u^1u^2u^3 - (u^1+u^2+u^3)(u^1u^2+u^2u^3+u^3u^1)}.$$

A lengthy but straightforward transformation of this operator into the original auxiliary variables yields the result in equation (17). The verification of the Jacobi identities for Hamiltonian operator (17) is straightforward. For this operator the conserved quantity $\frac{1}{2}a$ results in the trivial flow while c is a Casimir.

6. Variational principle

The most direct way of obtaining the second Hamiltonian structure of equation (8) is through the construction of a variational principle. The auxiliary variables a, b, c are not best suited for this purpose, instead it turns out that in terms of

$$p = f_x \qquad q = f_t \qquad r = f_{tt} \tag{24}$$

the Lagrangian is given by a simple local expression. The equations of motion are

$$p_t = q_x \qquad q_t = r \qquad r_t = \frac{r_x^2 - p_{xx}}{q_{xx}} \tag{25}$$

and it can be readily verified that the variational principle with the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}p_x p_t + q_x r_x q_t - \tfrac{1}{2}q_x^2 r_t + p q_{xx} + \tfrac{1}{2}r^2 q_{xx} \tag{26}$$

yields equations (25). We note that this Lagrangian is linear in the velocities so that the Hessian vanishes identically and we have a degenerate Lagrangian system. The passage to its Hamiltonian formulation requires the use of Dirac's theory of constraints [11].

7. Dirac bracket

The Dirac bracket which replaces the Poisson bracket for systems subject to constraints plays a central role in the construction of the Hamiltonian operator for integrable systems [18]. This construction is directly applicable to the WDVV equation (8) as we shall now detail. Since the Lagrangian (26) is degenerate the canonical momenta cannot be inverted for the velocities and following Dirac we introduce the definition of the momenta

$$\phi_1 = \pi_p - \tfrac{1}{2}p_x \qquad \phi_2 = \pi_q - q_x r_x \qquad \phi_3 = \pi_r + \tfrac{1}{2}q_x^2 \tag{27}$$

as primary constraints. Calculating the Poisson bracket of the constraints

$$\begin{aligned}
\{\phi_1(x), \phi_1(y)\} &= \tfrac{1}{2}\delta_x(y-x) - \tfrac{1}{2}\delta_y(x-y) \\
\{\phi_1(x), \phi_2(y)\} &= 0 \\
\{\phi_1(x), \phi_3(y)\} &= 0 \\
\{\phi_2(x), \phi_2(y)\} &= r_x\delta_x(y-x) - r_y\delta_y(x-y) \\
\{\phi_2(x), \phi_3(y)\} &= q_x\delta_x(y-x) + q_y\delta_y(x-y) \\
\{\phi_3(x), \phi_3(y)\} &= 0
\end{aligned} \tag{28}$$

we find that the constraints (27) are second class as they do not vanish modulo the constraints themselves. This is the case with almost all completely integrable systems. The total Hamiltonian density of Dirac is given by

$$\mathcal{H}_T = \mathcal{H}_1 + \sum_{i=1}^{3} c^i \phi_i \qquad \mathcal{H}_1 = -\tfrac{1}{2} r^2 q_{xx} - p q_{xx} \tag{29}$$

where c^i are Lagrange multipliers. The expression for $\mathcal{H}_1$ is obtained from the Lagrangian (26) by Legendre transformation. The conditions that the constraints are maintained in time $\{\phi_i(x), H_T\} = 0$ give rise to no further constraints but rather determine the Lagrange multipliers

$$c^1 = q_x \qquad c^2 = r \qquad c^3 = \frac{r_x^2 - p_{xx}}{q_{xx}}$$

and we have no secondary constraints. From equation (29) we find that the total Hamiltonian is given by

$$\mathcal{H}_T = q_x \pi_p + r \pi_q + \frac{r_x^2 - p_{xx}}{q_{xx}} \left(\pi_r + \frac{1}{2} q_x^2 \right) + \frac{1}{2} p_x q_x \tag{30}$$

in terms of the full set of canonical variables. For systems subject to second-class constraints we can solve the constraints to eliminate the canonical momenta because in Dirac's theory second-class constraints hold as strong equations. As a result the total Hamiltonian density is simply $\mathcal{H}_1$ which up to a total derivative is the same as $\mathcal{E}$ in equation (15).

Given any two smooth functionals A, B the Dirac bracket is defined by

$$\{A(x), B(y)\}_D = \{A(x), B(y)\} - \int \{A(x), \phi_i(z)\} J^{ik}(z, w) \{\phi_k(w), B(y)\} \, \mathrm{d}z \, \mathrm{d}w \tag{31}$$

where J^{ik} is the inverse of the matrix of Poisson brackets of the constraints. From the definition of the inverse

$$\int \{\phi_i(x), \phi_k(z)\} J^{kj}(z, y) \, \mathrm{d}z = \delta_i^j \, \delta(x - y)$$

we end up with a set of differential equations for J^{ik} which can be solved to yield

$$\begin{aligned}
J^{11}(x, y) &= -\theta(x - y) \\
J^{12}(x, y) &= 0 \\
J^{13}(x, y) &= 0 \\
J^{22}(x, y) &= 0 \\
J^{23}(x, y) &= -\frac{1}{q_{xx}} \delta(x - y) \\
J^{33}(x, y) &= -2 \frac{r_x}{q_{xx}^2} \delta_x(x - y) + \frac{r_x q_{xxx}}{q_{xx}^3} \delta(x - y)
\end{aligned} \tag{32}$$

where the Heaviside unit step function is denoted by θ.

8. Second Hamiltonian structure

The transition from the Dirac bracket to the Hamiltonian operator is given by

$$\{u^i(x), u^k(y)\}_D = -J^{ik}(x, y) \equiv -J^{ik}(x) \delta(x - y) \tag{33}$$

since the Poisson brackets of u^i vanish and the constraints are linear in the momenta. From equations (33) and (32) it follows that the Hamiltonian operator corresponding to the degenerate Lagrangian (26) is simply

$$J_1 = -\begin{pmatrix} D^{-1} & 0 & 0 \\ 0 & 0 & \frac{1}{q_{xx}} \\ 0 & -\frac{1}{q_{xx}} & \frac{1}{q_{xx}} D \frac{r_x}{q_{xx}} + \frac{r_x}{q_{xx}} D \frac{1}{q_{xx}} \end{pmatrix} \tag{34}$$

and the proof of the Jacobi identities for the Hamiltonian operator (34) follows from the fact that according to equation (33) it is simply a reformulation of the Dirac bracket for which there is a general proof of the Jacobi identities [19]. The Hamiltonian operator (34) looks non-local but this is only superficial and related to the choice of variables p, q and r. If we revert to the original auxiliary variables (13)

$$a = p_{xx} \qquad b = q_{xx} \qquad c = r_x$$

the Hamiltonian operator (34) is transformed to the form (18) which is a local homogeneous third-order operator of hydrodynamic type that was studied in [20, 21].

9. Symplectic representation

The symplectic formulation of equations (25) that provides the dual description to the Hamiltonian operator (34) requires the inverse of this operator. However, we have just seen that this Hamiltonian operator is derived from the Dirac bracket which in turn was obtained from the inverse of the Poisson bracket of the constraints. The matrix of the symplectic 2-form density can, therefore, be obtained directly from equations (28) and we get

$$\omega_{ij} = -\begin{pmatrix} D & 0 & 0 \\ 0 & Dr_x + r_x D & -q_{xx} \\ 0 & q_{xx} & 0 \end{pmatrix} \tag{35}$$

which can be verified to be the inverse of (34). The symplectic 2-form density is then given by

$$\omega = -\tfrac{1}{2}\,\mathrm{d}p \wedge \mathrm{d}p_x - r_x\,\mathrm{d}q \wedge \mathrm{d}q_x + q_{xx}\,\mathrm{d}q \wedge \mathrm{d}r \tag{36}$$

and since this 2-form is closed, using Poincaré's lemma we can write

$$\omega = \mathrm{d}\alpha \qquad \alpha = \tfrac{1}{2} p_x\,\mathrm{d}p - r q_{xx}\,\mathrm{d}q \tag{37}$$

in a local neighbourhood. In terms of f that enters into the original formulation of the WDVV equation (8) the symplectic 2-form (36) reduces to

$$\omega = 2 f_{tx}\,\mathrm{d}f_{tt} \wedge \mathrm{d}f_{tx} - \tfrac{1}{2}\,\mathrm{d}f_x \wedge \mathrm{d}f_{xx} \tag{38}$$

using the definitions (24) and discarding a total derivative.

10. Darboux's theorem

The transformation of the third-order Hamiltonian operator (18) to the canonical form

$$J_1 = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} D \tag{39}$$

required by an as yet unproved generalization of Darboux's theorem for third-order Hamiltonian operators is usually given by a differential substitution where the Casimirs of J_1 play a central role. For the Hamiltonian operator (18) the Casimirs are

$$s^1 = D^{-1}b \qquad s^2 = D^{-1}a \qquad s^3 = bD^{-1}c \tag{40}$$

and the differential substitution required by Darboux's theorem is obtained by inverting equations (40). Indeed it can be readily verified that under this transformation of variables the second Hamiltonian operator (18) is transformed to the form (39) which is proof of Darboux's theorem for this case. Finally, we note that in terms of the Casimir variables the equations of motion assume the form

$$\begin{aligned} s^1_t &= \left(\frac{s^3}{s^1_x}\right)_x \\ s^2_t &= s^1_x \\ s^3_t &= \left[\frac{s^3}{s^1_x}\left(\frac{s^3}{s^1_x}\right)_x - s^2\right]_x \end{aligned} \tag{41}$$

of an integrable coupled dispersive system.

11. Summary of earlier results

In order to compare the results on bi-Hamiltonian formulations of equations (8) and (6) we need to summarize the results of [5] on equation (6). In this case

$$e \equiv ac - b^2 \tag{42}$$

replaces the definition of the same quantity in equations (14). The Hamiltonian operators are given by

$$J'_0 = \begin{pmatrix} -\frac{3}{2}D & \frac{1}{2}Da & Db \\ \frac{1}{2}aD & \frac{1}{2}(Db + bD) & \frac{3}{2}cD + c_x \\ bD & \frac{3}{2}Dc - c_x & (b^2 - ac)D + D(b^2 - ac) \end{pmatrix} \tag{43}$$

$$J'_1 = \begin{pmatrix} 0 & 0 & D^3 \\ 0 & D^3 & -D^2aD \\ D^3 & -DaD^2 & D^2bD + DbD^2 + DaDaD \end{pmatrix} \tag{44}$$

and the corresponding densities of Hamiltonian functions

$$\mathcal{H}'_0 = c \qquad \mathcal{H}'_1 = -\tfrac{1}{2}a(D^{-1}b)^2 - (D^{-1}b)(D^{-1}c) \tag{45}$$

yield the system (14) with this redefinition of e. These Hamiltonian operators are also compatible. The symplectic 2-form obtained from the inverse of (44) is given by

$$\omega' = \mathrm{d}p \wedge \mathrm{d}r - q_{xx}\,\mathrm{d}p \wedge \mathrm{d}p_x + p_{xx}\,\mathrm{d}p \wedge \mathrm{d}q_x + \tfrac{1}{2}\,\mathrm{d}q \wedge \mathrm{d}q_x \tag{46}$$

which in terms of f in equation (6) is simply

$$\omega' = 2f_{tt}\,\mathrm{d}f_{tx} \wedge \mathrm{d}f_{tt} + \tfrac{3}{2}\,\mathrm{d}f_x \wedge \mathrm{d}f_{tx} \tag{47}$$

where we have again discarded a total derivative.

12. Covariant formulation

We have noted at the beginning that equations (8) and (6) are obtained by a simple flip of the independent variables. However, any comparison of the Hamiltonian operators (43) and (44) with (17) and (18) yields nothing more than a complete mismatch even though they arise from what is in fact the same WDVV equation. The fact that the Hamiltonian operators for these equations look very different forces us to look for a unifying framework which exists in the covariant formulation of symplectic structure [10]. Indeed these different looking Hamiltonian structures are simply different components of the Witten–Zuckerman closed, conserved current 2-form. In order to show that this is indeed the case we start with the Lagrangian for, say, equation (8)

$$\mathcal{L} = \tfrac{1}{2} f_{tt}^2 f_{txx} - f_{tx}^2 f_{ttt} - \tfrac{1}{2} f_{xx} f_{tx} \tag{48}$$

expressed in terms of the original variable f. This is the same as the Lagrangian (26) up to a total derivative. The Witten–Zuckerman current 2-form ω^μ which is closed and conserved

$$\omega_x^x + \omega_t^t = 0 \qquad \mathrm{d}\omega^\mu = 0 \qquad \mu = x, t \tag{49}$$

can be obtained directly from the Lagrangian (48). We find

$$\begin{aligned} \omega^x &= 2 f_{tt}\, \mathrm{d}f_{tx} \wedge \mathrm{d}f_{tt} + \tfrac{3}{2}\, \mathrm{d}f_x \wedge \mathrm{d}f_{tx} \\ \omega^t &= 2 f_{tx}\, \mathrm{d}f_{tt} \wedge \mathrm{d}f_{tx} - \tfrac{1}{2}\, \mathrm{d}f_x \wedge \mathrm{d}f_{xx} \end{aligned} \tag{50}$$

which satisfies the properties of the symplectic 2-form listed in equations (49). If we go back to equations (38) and (47) for the symplectic 2-forms appropriate to equations (8) and (6) expressed in terms of f alone, then we find simply

$$\omega^x = \omega' \qquad \omega^t = \omega \tag{51}$$

as we should expect, since these WDVV equations are related by a flip of t and x. Unfortunately, the lack of a variational formulation of WDVV equations (8) or (6) that yields their first Hamiltonian structure through Dirac's theory of constraints makes it impossible to present a similar covariant description of their first symplectic structure.

13. Second pair of WDVV equations

In order to discuss the Hamiltonian structure of the WDVV equation (11) we need to redefine

$$e \equiv \frac{b^2 + c^2 + bc - ac - a}{a + b} \tag{52}$$

in the system of evolution equations (14). The bi-Hamiltonian structure of these equations of hydrodynamic type is obtained through the same process we presented earlier and the result can be summarized in the following.

Theorem 2. The equations of hydrodynamic type (14) with e given by (52) admit bi-Hamiltonian structure (16) with Hamiltonian operators

$$\tilde{J}_0 = \begin{pmatrix} -aD - Da & \tfrac{1}{2} D(a-b) - bD & Db - cD \\ \tfrac{1}{2}(a-b)D - Db & \tfrac{1}{2}(b-c)D + \tfrac{1}{2} D(b-c) & \tfrac{1}{2}(c-e)D + Dc \\ bD - Dc & \tfrac{1}{2} D(c-e) + cD & \tfrac{3}{2} D + De + eD \end{pmatrix} \tag{53}$$

$$\tilde{J}_1 = \begin{pmatrix} -D^3 & D^3 & -D^3 \\ D^3 & -D^3 & D^2 \frac{a+b+1}{a+b} D \\ -D^3 & D\frac{a+b+1}{a+b}D^2 & D\frac{1}{a+b}D\frac{b+c+\frac{1}{2}}{a+b}D \\ & & +D\frac{b+c+\frac{1}{2}}{a+b}D\frac{1}{a+b}D \\ & & -D\frac{a+b+1}{a+b}D\frac{a+b+1}{a+b}D \end{pmatrix} \tag{54}$$

and the Hamiltonian functions which are integrals of the densities $\tilde{\mathcal{H}}_0 = c$ and

$$\tilde{\mathcal{H}}_1 = (c-b-a)D^{-1}bD^{-1}c + (b+1)D^{-1}aD^{-1}b + cD^{-1}aD^{-1}c + \tfrac{1}{2}(D^{-1}a)^2 \tag{55}$$

yield the equations of motion. The Hamiltonian operators $\tilde{J}_0$ and $\tilde{J}_1$ are compatible so that the theorem of Magri is applicable in this case as well.

Comparison of these results with the bi-Hamiltonian structure of equation (10) presented in [6] again results in a mismatch which, however, is not as severe as before since equations (10) and (11) resemble each other rather closely. In order to show that these different Hamiltonian structures simply correspond to different components of the Witten–Zuckerman 2-form first we note that the variational principle with the Lagrangian

$$\tilde{\mathcal{L}} = -\tfrac{1}{2}(f_{xx}^2 + f_{tx}f_{xx}) + f_{xxx}f_{tt}f_{xx} + f_{txx}f_{tx}f_{tt} - f_{xxx}f_{tt}f_{tx} - f_{ttt}f_{tx}f_{xx} - \tfrac{1}{2}f_{ttt}f_{tx}^2 \tag{56}$$

yields equation (11), or more precisely we get a linear combination of both the t and x derivatives of equation (11). This Lagrangian can be rewritten in terms of the auxiliary variables (24)

$$\begin{aligned} \tilde{\mathcal{L}} = {} & (-q_x q_{xx} + q_x r_x - r p_{xx} + p_x q_{xx} + \tfrac{1}{2}p_x)p_t - (\tfrac{1}{2}q_x^2 + p_x q_x)r_t \\ & +(q_x r_x - q_x p_{xx} + p_x p_{xx} + p_x r_x)q_t - q_x r r_x - \tfrac{1}{2}p_x^2 \\ & +q_x q_{xx} r + q_x p_{xx} r - p_x q_x q_{xx} - p_x r r_x - q_x p_x \end{aligned} \tag{57}$$

which yields a triplet of evolution equations, the integrability conditions of which result in equation (11).

The symplectic 2-form obtained from the inverse of the Hamiltonian operator (54) is given by

$$\tilde{\omega} = [(r_x + q_{xx})\,\mathrm{d}p_x + (r_x - p_{xx})\,\mathrm{d}q_x - (p_{xx} + q_{xx})\,\mathrm{d}r] \wedge (\mathrm{d}p + \mathrm{d}q) - \tfrac{1}{2}\,\mathrm{d}p \wedge \mathrm{d}p_x \tag{58}$$

and this is rather similar to the result

$$\tilde{\omega}' = [(r_x + q_{xx})\,\mathrm{d}p_x + (r_x - p_{xx})\,\mathrm{d}q_x - (1 + p_{xx} + q_{xx})\,\mathrm{d}r] \wedge (\mathrm{d}p + \mathrm{d}q) + \tfrac{1}{2}\,\mathrm{d}q \wedge \mathrm{d}q_x \tag{59}$$

for equation (10) that was obtained earlier in [6]. The Witten–Zuckerman symplectic 2-form that follows from the Lagrangian (56) is

$$\begin{aligned} \omega^x = {} & 2(f_{txx} - f_{ttx} - \tfrac{3}{4})\,\mathrm{d}f_{tx} \wedge \mathrm{d}f_x - (f_{ttx} + 1)\,\mathrm{d}f_{xx} \wedge \mathrm{d}f_x \\ & +(2f_{txx} - f_{xxx})\,\mathrm{d}f_{tt} \wedge \mathrm{d}f_x + (2f_{tt} + f_{xx})\,\mathrm{d}f_{xx} \wedge \mathrm{d}f_{tt} \\ & +2f_{tt}\,\mathrm{d}f_{tx} \wedge \mathrm{d}f_{tt} - f_{ttx}\,\mathrm{d}f_{tx} \wedge \mathrm{d}f_t + f_{txx}\,\mathrm{d}f_{tt} \wedge \mathrm{d}f_t - 2f_{tt}\,\mathrm{d}f_{tx} \wedge \mathrm{d}f_{xx} \\ = {} & \tilde{\omega}' \\ \omega^t = {} & \tfrac{1}{2}\,\mathrm{d}f_{xx} \wedge \mathrm{d}f_x + [(f_{ttx} + f_{txx})\,\mathrm{d}f_{xx} + (f_{ttx} - f_{xxx})\,\mathrm{d}f_{tx} \\ & -(f_{xxx} + f_{txx})\,\mathrm{d}f_{tt}] \wedge (\mathrm{d}f_x + \mathrm{d}f_t) \\ = {} & \tilde{\omega} \end{aligned} \tag{60}$$

where $\tilde{\omega}$, $\tilde{\omega}'$ are given by equations (59) and (58), respectively. This is exactly the same final result (51) we found earlier for equations (8) and (6).

14. Conclusion

The bi-Hamiltonian structure of the WDVV equations of associativity is expressed by different pairs of Hamiltonian operators depending on which independent variable in the free energy is chosen to play the role of time. However, we have shown that the symplectic 2-forms which can be obtained from the second Hamiltonian operators are simply different components of the covariant Witten–Zuckerman conserved current symplectic 2-form. We have been able to obtain this result only for the second Hamiltonian structure because only in this case does there exist a degenerate Lagrangian subject to second-class constraints which through the Dirac bracket yield the Hamiltonian operator. The lack of a variational principle for the WDVV equations that yields their different first Hamiltonian structures makes it impossible to discuss them together in the framework of the covariant theory of symplectic structure.

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